SPRING 2021: MATH 147 QUIZ 4 SOLUTIONS

Each question is worth 5 points. You must justify your answer to receive full credit.

1. Find the absolute maximum and absolute minimum values of $f(x, y) = x^2 + 2x + y^2 + 2y$ over the region $0 \leq x^2 + y^2 \leq 4$.

Solution. $f_x = 2x + 2$, $f_y = 2y + 2$. To find critical points in the interior of the region, we solve $2x + 2 = 0$ and $2y + 2 = 0$, so $x = -1, y = -1$. Thus, (-1,-1) is a critical point in the interior. On the boundary, we set $\nabla f = \lambda \nabla g$, for $g(x, y) = x^2 + y^2 = 4$. Upon doing so we get

$$2x + 2 = \lambda 2x$$
$$2y + 2 = \lambda 2y.$$

Note that neither x nor y can be zero. If we multiply the first y and the second equation by x and subtract, we get $2y - 2x = 0$, so $x = y$. From the constraint equation we have $x^2 + y^2 = 4$, so $x = \pm\sqrt{2}$. Thus the critical points on the boundary are: $(\sqrt{2}, \sqrt{2})$ and $-\sqrt{2}, -\sqrt{2})$. Testing the three critical points we get: $f(-1, -1) = -2$, $f(\sqrt{2}, \sqrt{2}) = 4+4\sqrt{2}$, $f(-\sqrt{2}, -\sqrt{2}) = 4-4\sqrt{2}$. Thus, -2 is the minimum value and $4+4\sqrt{2}$ is the maximum value.

2. Use a Lagrange multiplier to find the extreme value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x + y - z = 1$. Is this value a maximum or minimum?

Solution. If we write $g(x, y, z) = x + y - z = 1$, then upon setting $\nabla f = \lambda \nabla g$, we obtain,

$$2x = \lambda$$
$$2y = \lambda$$
$$2z = \lambda,$$

from which it follows that $2x = 2y = -2z$. Therefore, $y = x$ and $z = -x$. Using this in the constraint equation gives $x + y - (-x) = 1$, so $x = \frac{1}{3}$. It follows that the critical point is $(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3})$. The resulting extreme value of $f(x, y, z)$ subject to the given constraint is: $f(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$.

Notice that $\frac{1}{3}$ must be a minimum value of $f(x, y, z)$ subject to the constraint, since we can make $f(x, y, z)$ arbitrarily large over the constraint surface. For example, if we want $f(x, y, z) = N$, with N arbitrarily large, and x, y, z satisfying $x - y - z = 1$, take $z = 0, y = 1 - x$ and x satisfying $x^2 + (1 - x)^2 + 0^2 = N$. Note that such an x exists, since the equation $2x^2 - 2x + (1 - N) = 0$ has a real solution using the quadratic formula.

Comment: Though some students used the second derivative test for three variables, it does not apply to the situation where the extreme values are sought under a constraint.